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Propagation of shock waves in a dusty gas with exponentially varying density

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Abstract. The variation of flow-variables with distance, in the flow-field behind a shock wave propagating in a dusty gas with exponentially varying density, are obtained at different times. The equilibrium flow conditions are assumed to be maintained, and the results are compared with those obtained for a perfect gas. It is found that the presence of small solid particles in the medium has significant effects on the variation of density and pressure.

PACS. 47.40.-x Compressible flows; shock and detonation phenomena – 47.55.Kf Multiphase and particle-laden flows

1 Introduction

The study of high speed flow of a mixture of gas and small solid particles is of great interest in several branches of engineering and science (Pai *et al.* [1]). The propagation of strong shock wave produced on account of sudden explosion in a medium where the density varies as some power of the distance from the point of explosion, has been studied by Christer and Helliwell [2], Verma [3] and many others. Hayes [4], Ray and Bhowmick [5], Verma and Vishwakarma [6] have studied the propagation of plane shock wave in a medium where density increases exponentially.

In our study, we have generalized the solution of Ray and Bhowmick [5] in gas to the case of two phase flow of a mixture of gas and small solid particles in which the density obeys the exponential law. In order to get some essential features of shock propagation, small solid particles are considered as a pseudo-fluid, and it is assumed that the equilibrium flow condition is maintained in the flow field, and that the viscous stress and heat conduction of the mixture are negligible [1]. Although the density of the mixture is assumed to be increasing exponentially, the volume occupied by the solid particles may be very small under ordinary conditions owing to the large density of the particle material. Hence for simplicity the initial volume fraction of solid particles Z_1 is assumed to be a small constant. Our solutions obtained are non-similar ones and are valid for the time till Z_1 remains small. Variation of the flow variables with distance, behind the shock front, at different times, are shown in Figures.

2 Fundamental equations and boundary conditions

The fundamental equations for one dimensional and unsteady flow of a mixture of gas and small solid particles can be written as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0 \qquad (2.1)$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + \rho\frac{\partial u}{\partial r} + \frac{i\rho u}{r} = 0 \qquad (2.2)$$

$$\frac{\partial U_\mathrm{m}}{\partial t} + u\frac{\partial U_\mathrm{m}}{\partial r} - \frac{p}{\varrho^2}\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r}\right) = 0 \qquad (2.3)$$

where $i = 0, 1$ or 2 correspond to plane, cylindrical or spherical symmetry,

ρ	is the density of mixture,
u	the flow velocity,
p	the pressure of mixture,
U_m	the internal energy per unit mass of the mixture,
r	the distance, and
t	the time.

The equation of state of the mixture of gas and small solid particles can be written as (Pai *et al.* [1])

$$p = \frac{(1-k_\mathrm{p})}{(1-Z)}\,\rho\,R^*T \qquad (2.4)$$

where R^* is the gas constant, T the temperature, k_p the mass concentration of solid particles and Z the volume fraction of solid particles in the mixture.

The relation between k_p and Z is given by

$$k_p = \frac{Z\,\rho_{sp}}{\rho} \qquad (2.5)$$

where ρ_{sp} is the species density of solid particles. In equilibrium flow, k_p is a constant in the whole flow field.

The internal energy of the mixture may be written as follows

$$U_m = k_p\,C_{sp} + (1-k_p)C_v = C_{vm}\,T, \qquad (2.6)$$

where C_{sp} is the specific heat of solid particles, C_v the specific heat of gas at constant volume, and C_{vm} the specific heat of the mixture at constant volume. The specific heat at constant pressure process is

$$C_{pm} = k_p C_{sp} + (1-k_p)C_p \tag{2.7}$$

where C_p is the specific heat of the gas at constant pressure process.

The ratio of the specific heats of the mixture is given by (Marble [7], Pai *et al.* [1])

$$\Gamma = \frac{C_{pm}}{C_{vm}} = \gamma \frac{(1+\delta\beta'/\gamma)}{1+\delta\beta'}, \tag{2.8}$$

where $\gamma = \frac{C_p}{C_v}$, $\delta = \frac{k_p}{1-k_p}$, and $\beta' = \frac{C_{sp}}{C_v}$.

The internal energy is therefore, given by

$$U_{\mathrm{m}} = \frac{p(1-Z)}{\rho(\Gamma-1)}\cdot \tag{2.9}$$

We consider that a strong shock wave is propagated into a medium, at rest, with negligibly small counter pressure. Also the initial density of the medium (the mixture of a gas and small solid particles) is assumed to obey the exponential law,

$$\rho = k\mathrm{e}^{\alpha r}, \tag{2.10}$$

where α and k are positive constants.

The jump conditions across the shock wave are as

$$\begin{aligned} u_2 &= (1-\beta)U, \\ \rho_2 &= \frac{\rho_1}{\beta}, \\ p_2 &= (1-\beta)\rho_1 U^2, \\ Z_2 &= \frac{Z_1}{\beta}, \end{aligned} \tag{2.11}$$

where suffices "1" and "2" refer to the values just ahead and just behind of the shock, $U = \frac{\mathrm{d}R}{\mathrm{d}t}$ is the shock velocity, and R the distance of the shock front from the plane, the line or the point of symmetry. Also the quantity "β" is given by

$$\beta = \frac{\Gamma + 2Z_1 - 1}{\Gamma+1}\cdot \tag{2.12}$$

The initial volume fraction of the solid particles Z_1 is, in general not a constant. But the volume occupied by the solid particles is very small because the density of the solid particles is much larger than that of the gas (Miura and Glass [8]), hence Z_1 may be assumed as a small constant. The expression for Z_1 is (Naidu *et al.* [9])

$$Z_1 = \frac{k_p}{G(1-k_p)+k_p} \tag{2.13}$$

where $G = \frac{\rho_{sp}}{\rho_g}$ the ratio of the density of solid particles to the density of gas. Values of Z_1 for some typical values of k_p and G are given in Table 1.

Table 1. Values of Z_1 for some typical values of k_p and G.

k_p	G	Z_1
	50	0.00222
0.1	100	0.00111
	200	0.00056
	50	0.00498
0.2	100	0.00249
	200	0.00125
	50	0.01316
0.4	100	0.00662
	200	0.00332

Let the solution of equations (2.1), (2.2) and (2.3) be of the form,

$$\begin{aligned} u &= t^{-1}V(\eta), \\ \rho &= t^{\Omega}D(\eta), \\ p &= t^{\Omega-1}H(\eta), \end{aligned} \tag{2.14}$$

where

$$\eta = t\mathrm{e}^{\lambda x} \quad , \quad \lambda \neq 0 \tag{2.15}$$

and the constants Ω and λ are to be determined subsequently. We choose the shock surface to be given by

$$\eta_0 = \text{const.} \tag{2.16}$$

so that its velocity is given by

$$U = -\frac{1}{\lambda t}\cdot \tag{2.17}$$

Hence it is obvious that $\lambda < 0$. The solutions of the equations (2.1) to (2.3) in the form of (2.14) are compatible with the shock conditions only if,

$$\Omega = 2 \qquad \text{and} \qquad \lambda = -\frac{\alpha}{2}, \tag{2.18}$$

The effective shock Mach number M_{e} is given by

$$M_{\mathrm{e}}^2 = \frac{U^2}{a_1^2} = \frac{U^2}{[p_1/\rho_1(1-Z_1)]} = \frac{4(1-Z_1)k}{\Gamma p_1 \alpha^2 \eta_0^2} = \text{const.}$$

Since M_{e} comes out to be a constant and p_1 can be taken to be of order zero for a very strong shock, we conclude that the shock retains its great strength even for a large time. Hence our solutions obtained in the next section are applicable for any time $t > \tau$ till Z_1 remains small, τ being the duration of initial impulse.

Also, from equation (2.17) and (2.18), we obtain

$$R = \frac{2}{\alpha}\log\frac{t}{\tau}\cdot \tag{2.19}$$

3 Solution of the equations

The flow variables inside the shock wave are obtained by solving the equations (2.1), (2.2) and (2.3). From equations (2.14), (2.17) and (2.18), we obtain

$$\frac{\partial\rho}{\partial t}=U\alpha\rho-U\frac{\partial\rho}{\partial r}, \quad (3.1)$$

$$\frac{\partial p}{\partial t}=-U\frac{\partial p}{\partial r}, \quad (3.2)$$

$$\frac{\partial u}{\partial t}=uU\lambda-U\frac{\partial u}{\partial r}\cdot \quad (3.3)$$

Using equations (3.1-3.3) and the transformations

$$r'=\frac{r}{R}\,,\qquad u'=\frac{u}{U}\,,\qquad p'=\frac{p}{p_2},\qquad \rho'=\frac{\rho}{\rho_2} \quad (3.4)$$

in the fundamental equations (2.1-2.3), we obtain

$$\frac{\mathrm{d}u'}{\mathrm{d}r'}=\frac{1}{1-u'}\,\rho U^2\frac{\mathrm{d}p'}{\mathrm{d}r'}-\frac{u'\alpha R}{2(1-u')}, \quad (3.5)$$

$$\frac{1}{\rho'}\frac{\mathrm{d}\rho'}{\mathrm{d}r'}=\frac{\alpha R}{1-u'}+\frac{1}{1-u'}\frac{\mathrm{d}u'}{\mathrm{d}r'}+\frac{iu'}{(1-u')r'}, \quad (3.6)$$

$$\frac{1}{p'}\frac{\mathrm{d}p'}{\mathrm{d}r'}=\frac{\Gamma}{(1-Z)\rho'}\frac{\mathrm{d}\rho'}{\mathrm{d}r'}-\frac{\Gamma\alpha R}{(1-Z)(1-u')}\cdot \quad (3.7)$$

From above equations, we have

$$\frac{\mathrm{d}p'}{\mathrm{d}r'}=\frac{\rho'p'\Gamma\beta}{2r'}\left[\frac{2iu'(1-u')-\alpha Ru'r'}{\rho'(\beta-Z_1\rho')(1-u')^2-\Gamma\beta^2p'(1-\beta)}\right], \quad (3.8)$$

$$\frac{\mathrm{d}u'}{\mathrm{d}r'}=\frac{p'\Gamma\beta^2(1-\beta)}{2r'(1-u')}\times\left[\frac{2iu'(1-u')-\alpha Ru'r'}{\rho'(\beta-Z_1\rho')(1-u')^2-\Gamma\beta^2p'(1-\beta)}\right]-\frac{u'\alpha R}{2(1-u')}, \quad (3.9)$$

$$\frac{\mathrm{d}\rho'}{\mathrm{d}r'}=\frac{\alpha R\rho'}{1-u'}+\frac{\rho'p'\Gamma\beta^2(1-\beta)}{2r'(1-u')^2}\times\left[\frac{2iu'(1-u')-\alpha Ru'r'}{\rho'(\beta-Z_1\rho')(1-u')^2-\Gamma\beta^2p'(1-\beta)}\right]-\frac{u'\alpha R\rho'}{2(1-u')^2}+\frac{iu'\rho'}{(1-u')r'}\cdot \quad (3.10)$$

Also, the total energy of the flow field behind the shock front is given by

$$E=\sigma_i\int_0^R\rho(U_m+\frac{1}{2}u^2)r^i\mathrm{d}r, \quad (3.11)$$

where $\sigma_i=2\pi i+(i-1)(i-2)$, using (3.4), (3.11) becomes

$$E=\frac{\sigma_i\rho_1U^2R^{i+1}}{\beta}\int_0^1\left[\frac{1}{2}\rho'u'^2+\frac{(\beta-Z_1\rho')(1-\beta)p'}{(\Gamma-1)}\right](r')^i\mathrm{d}r'$$
$$=\frac{4k\sigma_iR^{i+1}}{\beta\alpha^2\tau^2}\int_0^1\left[\frac{1}{2}\rho'u'^2+\frac{(\beta-Z_1\rho')(1-\beta)p'}{(\Gamma-1)}\right](r')^i\mathrm{d}r'.$$

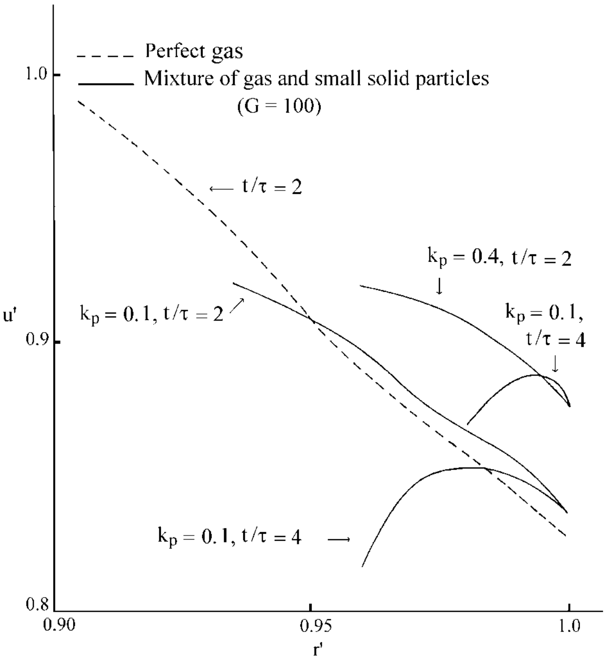

Fig. 1. Variation of reduced flow velocity u' in the region behind the shock front.

Hence the total energy of the shock wave is non-constant and varies as R^{i+1}, where $i=0,1$ or 2 for plane, cylindrical or spherical shock.

In terms of dimensionless variables r', p', ρ' and u' the shock conditions take the form

$$r'=1\,,\qquad p'=1\,,\qquad \rho'=1\,,\qquad u'=1-\beta\cdot \quad (3.12)$$

Equations (3.8) to (3.10) along with the boundary conditions (3.12) give the solution of our problem. The solution thus obtained is a non-similar one, since the motion behind the shock can be determined only when a definite value for time is prescribed.

4 Results and discussion

To obtain the solutions, we start numerical integration of the equations (3.8) to (3.10) from the shock front ($r'=1$) and proceed inwards. Distributions of flow variables $u'=\frac{u}{U}$, $p'=\frac{p}{p_2}$, $\rho'=\frac{\rho}{\rho_2}$ are obtained for spherical shock ($i=2$) at given instants for which $t/\tau=2$ or 4. Values of γ, $k_{p'}$ G and β' are taken as $\gamma=1.4$, $k_p=0.1$, 0.4, $G=100$ (see Pai *et al.* [1]), $\beta'=1$ (see Miura and Glass [8]) and the solutions are shown in Figures 1 to 3.

Figure 1 shows that in the initial stages of motion ($t/\tau=2$), the velocity u' increases from shock front to the inner contact surface, but at latter stages ($t/\tau=4$) it decreases after attaining a maximum. Also, it is shown that for small values of k_p, the values of u' tend to be those for the corresponding perfect gas as indicated in the work of Pai *et al.* [1].

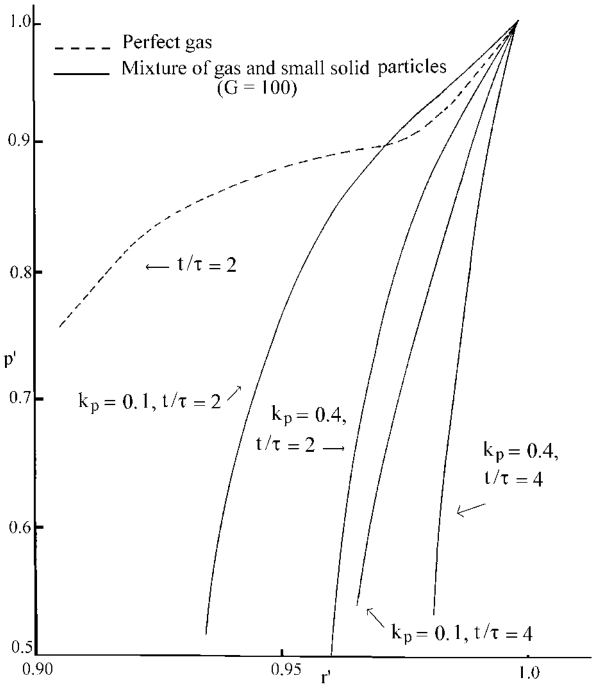

Fig. 2. Variation of reduced pressure p' in the region behind the shock front.

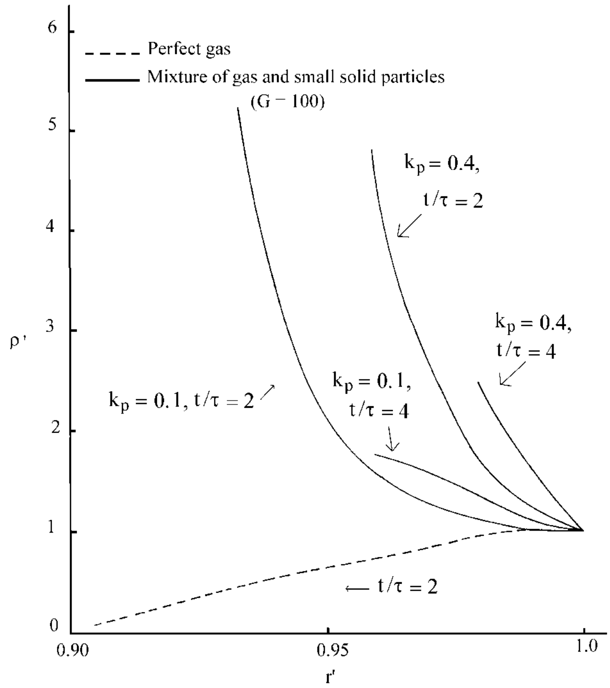

Fig. 3. Variation of reduced density ρ' in the region behind the shock front.

Figure 2 shows that the pressure p' decreases as we move inwards from the shock front. This decrease of pressure is faster for dusty gas in comparison to that for the corresponding perfect gas.

Figure 3 shows that the density ρ' increases as we move from shock front to the inner contact surface which is in contrast with the case of perfect gas (Verma and Singh [10], Ray and Bhowmick [5]), where it decreases. This phenomenological behaviour of the density is attributed to the presence of solid particles in the dusty gas. Actually, in the case of a strong shock in a perfect gas, the most of the mass is concentrated near the shock front, and follows it. On the other hand, in the case of a dusty gas the shock speed is reduced relative to the inner contact surface (as indicated in the next paragraph), and the medium has the tendency to stagnate due to load of solid particles. Therefore the region behind the shock front is driven by the inner contact surface causing the increase of density in its neighbourhood.

Figures 1-3 show that the effects of an increase in the mass concentration of solid particles k_p are:

(i) to increase the velocity u',
(ii) to increase the slopes of the pressure and density profiles in the region behind the shock front, and
(iii) to decrease the distance between the inner contact surface and the shock front. This results due to the fact that the shock speed is reduced when shock moves in a dusty gas (when compared with perfect gas) or in a dusty gas with comparatively higher k_p.

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